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BPS QUANTIZATION OF THE FIVE-BRANE

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ABSTRACT

We give a unified description of all BPS states of M-theory compactified on T^5 in terms of the five-brane. We compute the mass spectrum and degeneracies and find that the $SO(5, 5, \mathbf{Z})$ U-duality symmetry naturally arises as a T-duality by assuming that the world-volume theory of the five-brane itself is described by a string theory. We also consider the compactification on $S^1/\mathbf{Z}_2 \times T^4$, and give a new explanation for its correspondence with heterotic string theory by exhibiting its dual equivalence to M-theory on $K3 \times S^1$.

1. INTRODUCTION

There is by now considerable evidence that the various dual relations between different string theories can eventually be understood in terms of an underlying “M-theory”, whose low energy effective action is given by eleven dimensional supergravity [1, 2, 3, 4, 5, 6, 7, 8, 9], or its twelve-dimensional variant [10, 11]. It is not yet known what M-theory looks like, but it is reasonable to expect that just like string theory it has some formulation in terms of fluctuating extended objects, being the membrane or its dual, the five-brane. The fundamental strings that we know in ten dimensions or less are obtained by dimensional reductions from the membrane or the five-brane. From this point of view these branes must be considered just as fundamental as strings [4]. In particular, one expects that upon quantization their spectrum will take the form of a tower of states in an analogous fashion as for strings.

Unlike string theory, M-theory does not have a perturbative coupling constant, since there is no dilaton-field in 11-dimensional supergravity. The dual relationship between membranes and five-branes must therefore be different from more standard weak-strong coupling dualities, such as between strings and five-branes in $d = 10$. In particular, there is the logical possibility of a double correspondence, in which the dual membrane may in fact also be viewed as a particular limiting configuration of the five-brane itself. Hence M-theory could in a sense be self-dual. This possibility is supported by the fact that, besides to the dual six-form \tilde{C}_6 of eleven-dimensional supergravity, the five-brane also couples directly to the three-form C_3 itself, via an interaction of the type

$$\int C_3 \wedge T_3, \tag{1.1}$$

where T_3 is a self-dual three-form field strength that lives on the world-volume. By allowing this field T_3 to have non-trivial fluxes through the three-cycles on the world-brane, the five-brane can thus in principle carry all membrane quantum numbers. These configurations are therefore naturally interpreted as bound states between the two types of branes.

One of the aims of the eleven-dimensional point of view is to shed light on the mysterious U-duality symmetry of string theory [1]. The most convincing evidence for U-duality so far has been obtained by considering the spectrum of BPS states. These studies necessarily involve D-branes that describe the states charged with respect to the RR gauge fields [12]. Indeed, there have been convincing results in D-brane analysis, in particular in the form of degeneracy formulas, that support the symmetry under certain U-duality transformations [13, 14]. However, the formalism quickly becomes rather involved, since in general one has to take into account D-branes of various dimensions and also bound states of fundamental strings and D-branes [15].

As we just argued, the eleven-dimensional five-brane is a natural candidate to give a more unified treatment of all BPS states in string theory. In fact, one could hope that, in an appropriate covariant quantization, U-duality becomes a manifest symmetry of the five-brane. Unfortunately, there are various difficulties in extending the covariant formalism from strings to higher-dimensional extended objects [16]. In this paper we will make a first step in developing a formalism for describing the BPS configurations of the five-brane for compactifications down to six dimensions, which indeed exhibits the maximal symmetry. An important ingredient in this formalism is the idea that the relevant degrees of freedom on the five-brane are formed by the ground states of a string theory living on the world-volume itself. In fact, in compactifications on a 5-torus the U-duality group $SO(5, 5, \mathbf{Z})$ can then be identified with the T-duality group of this string.

In six-dimensional compactifications only the ground states of this string give rise to space-time BPS states. The string excitations appear only after further compactification down to five and four dimensions, where one can consider BPS representations that are annihilated by $1/8$ instead of $1/4$ of the supercharges. The structure of the resulting BPS spectrum has been described in our previous paper [17]. In this paper we will restrict ourselves to the six-dimensional theory.

OUTLINE OF THE PAPER

In section 2 we will start with a detailed description of the BPS spectrum of M-theory in six dimensions. First we derive the BPS mass formula from the space-time supersymmetry algebra. Then, by comparing with the known result of the BPS spectrum of type IIA strings, we propose an explicit formula for all the degeneracies that is manifestly invariant under the complete U-duality group $SO(5, 5, \mathbf{Z})$. In section 3 we analyze the zero-mode structure of the five-brane and show that the central charges correspond to particular fluxes through homology cycles on the five-brane. We use this in section 4 to construct the complete space-time supersymmetry algebra, including the central charges, as operators in the Hilbert space of the five-brane. In section 5 we first argue that U-duality implies that the world-brane theory must contain string degrees of freedom. We then describe a possible light-cone formulation of this six-dimensional string theory. The low-energy fields will correspond to the collective modes of the five-brane. Finally, in section 6 we set out to calculate the BPS spectrum of the five-brane, first by considering the linearized quantum fluctuations and then by including the winding quantum numbers. We also discuss the relation with more conventional D-brane counting. In section 7 we consider M-theory compactifications on the orbifolds $S^1 \times K3$ and $T^4 \times S^1/\mathbf{Z}_2$. Here we find a new derivation of the correspondence with heterotic string theory and demonstrate the dual equivalence of these two compactifications.

2. THE BPS MASS FORMULA AND U-DUALITY

Before we turn to our discussion of the five-brane theory, let us first give a description of the BPS states and the mass formula from the six dimensional space-time point of view. The maximally extended six-dimensional $N = (4, 4)$ supersymmetry algebra is given by [18]

$$\begin{aligned}\{Q_\alpha^a, Q_\beta^b\} &= \omega^{ab} \gamma^\mu \not{p}_{\alpha\beta}, \\ \{Q_\alpha^a, \overline{Q}_\beta^b\} &= \delta_{\alpha\beta} Z^{ab},\end{aligned}\tag{2.1}$$

where $a, b = 1, \dots, 4$ are $SO(5)$ spinor indices and ω^{ab} is an anti-symmetric matrix, that will be used to raise and lower indices. (We refer to the Appendix for our conventions on spinors and gamma-matrices.) The algebra contains 16 central charges that are combined in the 4×4 matrix Z^{ab} and transform as a bi-spinor under the $SO(5) \times SO(5)$ R-symmetry. It further satisfies the reality condition $Z^* = \omega Z \omega^T$. We now wish to obtain a convenient expression of the masses of BPS states in terms of the matrix Z^{ab} .

In general, BPS states form short multiplets of the supersymmetry algebra which are annihilated by a subset of the supersymmetry generators. We will see that in six dimensions the generic BPS state is annihilated by 1/4 of the 32 supercharges. For a given multiplet the condition can be written as

$$(\varepsilon_a Q^a + \overline{\varepsilon}_b \overline{Q}^b)|\text{BPS}\rangle = 0.\tag{2.2}$$

Since this condition holds with fixed $\varepsilon, \overline{\varepsilon}$ for all states in the BPS multiplet, we can take the commutator with the supercharges, and derive the following conditions on the supersymmetry parameters

$$\begin{aligned}\not{p} \varepsilon^a + Z^{ab} \overline{\varepsilon}_b &= 0, \\ \not{p} \overline{\varepsilon}_a + Z_{ab}^\dagger \varepsilon^b &= 0.\end{aligned}\tag{2.3}$$

Combining these equations with the mass shell condition $p^2 + m_{\text{BPS}}^2 = 0$, one deduces that m_{BPS}^2 coincides with the highest eigenvalue of the hermitean matrices ZZ^\dagger and $Z^\dagger Z$, with ε and $\overline{\varepsilon}$ being the corresponding eigenvectors,

$$\begin{aligned}(ZZ^\dagger)^a{}_b \varepsilon^b &= m_{\text{BPS}}^2 \varepsilon^a, \\ (Z^\dagger Z)^a{}_b \overline{\varepsilon}_b &= m_{\text{BPS}}^2 \overline{\varepsilon}_a.\end{aligned}\tag{2.4}$$

This determines the BPS masses completely in terms of the central Z^{ab} .

The number of states within a BPS supermultiplet are determined by the number of eigenvectors with the highest eigenvalue m_{BPS}^2 . In fact, one can show that for a given (non-zero) value of m_{BPS} there are always two independent eigenvectors ε and $\bar{\varepsilon}$. This can be seen, for example, by decomposing the matrices ZZ^\dagger and $Z^\dagger Z$ in terms of hermitean gamma-matrices Γ_m , $m = 1, \dots, 5$ as

$$\begin{aligned} ZZ^\dagger &= m_0^2 \mathbf{1} + 2K_L^m \Gamma_m, \\ Z^\dagger Z &= m_0^2 \mathbf{1} + 2K_R^m \Gamma_m. \end{aligned} \quad (2.5)$$

These relations, which define m_0^2 and the 5-vectors K_L and K_R , directly follow from the reality condition on Z^{ab} . The eigenvalue equation for the spinors ε and $\bar{\varepsilon}$ are now replaced by the Dirac-like conditions

$$\begin{aligned} (K_L \cdot \Gamma - |K_L|) \varepsilon &= 0, \\ (K_R \cdot \Gamma - |K_R|) \bar{\varepsilon} &= 0, \end{aligned} \quad (2.6)$$

which have indeed (generically) two independent solutions. Thus the BPS condition (2.2) can be imposed for 8 of the 32 supercharges. Consequently, a BPS supermultiplet contains $(16)^3$ states: this is in between the size of a massless multiplet, which has $(16)^2$ states, and that of a generic massive supermultiplet with $(16)^4$ states.

The ten components (K_L, K_R) can be understood as follows. We have seen that the 16 components of the central charge can naturally be combined into a spinor Z of $SO(5, 5)$. Out of two such spinors we can construct in the usual way a ten-dimensional vector with components $K_L^m = \frac{1}{8} \text{tr}(\Gamma^m Z Z^\dagger)$ and $K_R^m = \frac{1}{8} \text{tr}(\Gamma^m Z^\dagger Z)$ as introduced in (2.5). In fact, these quantities form a *null* vector (K_L, K_R) of $SO(5, 5)$, since one easily verifies that $|K_L| = |K_R|$. We can express the BPS masses in terms of this vector, by combining the results (2.4) and (2.5). We find

$$m_{\text{BPS}}^2 = m_0^2 + 2|K_{L,R}|, \quad (2.7)$$

where $m_0^2 = \frac{1}{4} \text{tr}(ZZ^\dagger)$. The above BPS mass formula is invariant under an $SO(5) \times SO(5)$ symmetry, which acts on Z_{ab} on the left and on the right respectively.

U-DUALITY INVARIANT MULTIPLICITIES OF BPS STATES

Charge quantization implies that the central charge Z^{ab} is a linear combination of integral charges. The expression of Z^{ab} in terms of the integers depends on the expectation values of the 25 scalar fields of the 6-dimensional $N = (4, 4)$ supergravity theory. From the point of view of eleven dimensions these scalars represent the metric G_{mn}

and three-form C_{mnk} on the internal manifold T^5 , and parametrize the coset manifold $\mathcal{M} = SO(5, 5)/SO(5) \times SO(5)$. A convenient way to parametrize this coset is to replace the 3-form C_3 on T^5 by its Hodge-dual $B = *C_3$. The parametrization of \mathcal{M} in terms of G_{mn} and B_{mn} is then familiar from toroidal compactifications in string theory. Infinitesimal variations of G and B are represented via the action of the spinor representation of $SO(5, 5)$ on Z^{ab} . Concretely,

$$\delta Z^{ab} = (\delta G^{mn} + \delta B^{mn})(\Gamma_m Z \Gamma_n)^{ab}, \quad (2.8)$$

where Γ_m are hermitean gamma-matrices of $SO(5)$. The U-duality group is now defined as those $SO(5, 5)$ rotations that map the lattice of integral charges on to itself. Thus the U-duality group can be identified with $SO(5, 5, \mathbf{Z})$.

The 16 charges contained in Z_{ab} can be interpreted in various ways depending on the starting point that one chooses. In this paper we will be interested in the BPS states that come from the five-brane in 11 dimensions. From the point of view of the five-brane it is natural to break the $SO(5, 5, \mathbf{Z})$ to a $SL(5, \mathbf{Z})$ subgroup, because this represents the mapping-class group of the five-torus T^5 . The 16 charges split up in 5 Kaluza-Klein momenta r^m , 10 charges s_{mn} that couple to the gauge-fields C_μ^{mn} that come from the 3-form, and one single charge q that represents the winding number of the five-brane. We can work out the BPS mass formula in terms of these charges q , r^m and s_{mn} . To simplify the formula we consider the special case where the scalar fields associated with three-form C_{mnk} are put to zero, and the volume of T^5 is put equal to one. Note that these restrictions are consistent with the $SL(5)$ symmetry. For this situation the central charge Z_{ab} takes the form

$$Z_{ab} = q \mathbf{1}_{ab} + r^m \Gamma_{m,ab} + s_{mn} \Gamma_{ab}^{mn}, \quad (2.9)$$

where the Dirac-matrices satisfy $\{\Gamma_m, \Gamma_n\} = 2G_{mn}$. (Note that in our notation $\mathbf{1}_{ab} = \omega_{ab}$.) Inserting this expression in to the BPS mass formula gives

$$m_{\text{BPS}}^2 = m_0^2 + 2\sqrt{G_{mn}K^m K^n + G^{mn}W_m W_n}, \quad (2.10)$$

with

$$\begin{aligned} m_0^2 &= q^2 + G_{mn}r^m r^n + G^{mk}G^{nr}s_{mn}s_{kr}, \\ K^m &= q r^m + \frac{1}{2}\epsilon^{mnklr}s_{nk}s_{lr}, \\ W_m &= s_{mn}r^n. \end{aligned} \quad (2.11)$$

Here we have written $K_{L,R}^m = K^m \pm G^{mn} W_n$.

The BPS spectrum has the following interpretation in terms of the toroidal compactification of the type IIA string. Because the string coupling constant coincides with one of the metric-components, say G_{55} , string perturbation theory breaks the $SO(5, 5, \mathbf{Z})$ U-duality to a manifest $SO(4, 4, \mathbf{Z})$ T-duality. Accordingly, the 16 charges split up in an $SO(4, 4)$ vector of NS-charges, being the 4 momenta and 4 string winding numbers n^i ($= r^i$), and m_i ($= s_{i5}$), and an 8 dimensional spinor that combines the RR-charges q , s_{ij} and r ($= r_5$) of the 0-branes, 2-branes and 4-branes. U-duality relates RR-solitons to the perturbative string states, and can thus be used to predict the multiplicities of the solitonic BPS states from the known spectrum of string BPS states. This fact has been exploited by Sen and Vafa [14, 13] to give a non-trivial check on U-duality by reproducing the expected multiplicities from the D-brane description of the RR-solitons [12]. The multiplicities of the string BPS states, *i.e.* with vanishing RR-charges, is given by $d(N)$ where $N = n^i m_i$ and

$$\sum_N d(N) t^N = (16)^2 \prod_k \left(\frac{1 + t^k}{1 - t^k} \right)^8. \quad (2.12)$$

This formula also describes the degeneracies of the RR-solitons where $N = qr + \frac{1}{2} s_{ij} \tilde{s}^{ij}$ represents the self-intersection number of the D-branes. In fact, with the help of our analysis, it is not difficult to obtain a generalized formula that satisfies all requirements: we find that the degeneracies are given by the same numbers $d(N)$, but where N is now given by the greatest common divisor of the ten integers K^m and W_m ,

$$N = \text{gcd}(K^m, W_m). \quad (2.13)$$

Indeed, one easily verifies that for the string BPS states all integers K^m and W_m vanish except $W_5 = n^i m_i$, while for the RR-solitons the only non-vanishing component is $K^5 = qr + \frac{1}{2} s_{ij} \tilde{s}^{ij}$. Furthermore, the formula is clearly invariant under the U-duality group $SO(5, 5, \mathbf{Z})$. It can be shown that this is the unique degeneracy formula with all these properties.*

In the remainder of this paper we will present evidence that *all* BPS states can be obtained from the five-brane. To this end we will propose a concrete quantum description of the five-brane dynamics that reproduces the complete BPS spectrum, including the above degeneracy formula.

*This follows from the fact that any two primitive null vectors $v, w \in \Gamma^{5,5}$ can be rotated into each other by a $SO(5, 5, \mathbf{Z})$ transformation. This is implied by the isomorphism $v^\perp / \langle v \rangle \cong \Gamma^{4,4}$, *i.e.* the little group of v is always $SO(4, 4, \mathbf{Z})$. We thank G. Moore and C. Vafa for discussions on this point.

3. CHARGES AND FLUXES ON THE FIVE-BRANE

In this section we consider the compactification of the five-brane coupled to eleven-dimensional supergravity to six dimensions on a five-torus T^5 . From its description as a soliton it is known that, after appropriate gauge-fixing, the five-brane is described by an *effective* world-brane theory consisting of five scalars, an anti-symmetric tensor with self-dual field strength $T = dU$ and 4 chiral fermions [19]. These fields, which parametrize the collective modes of the five-brane solution, form a tensor multiplet of the chiral $N = (4, 0)$ supersymmetry on the world-brane.

The five-brane couples directly to the six-form \tilde{C}_6 , the metric and the three-form C_3 , and hence after dimensional reduction it is charged with respect to all the corresponding 16 gauge fields that we denote as A, A^m and A^{mn} . To make this concrete, we consider a five-brane with topology of $T^5 \times \mathbf{R}$ where \mathbf{R} represents the world-brane time τ . First we consider the coupling to A^{mn} , which is deduced from the term (1.1) by taking $C_3 = A_{mn} \wedge dX^m \wedge dX^n$, where X^m are the embedding coordinates of the five-brane, and A^{mn} is constant along T^5 . Now we use that $dX^m \wedge dX^n$ represents a closed two-form, and hence defines a dual three-cycle T_{mn}^3 . In this way we find that the charge s_{mn} with respect to the gauge fields A^{mn} is given by the flux of the self-dual three-form field strength $T = dU$

$$s_{mn} = \int_{T_{mn}^3} dU \quad (3.1)$$

through the 10 three-cycles T_{mn}^3 on the five-brane. A five-brane for which these charges are non-zero, is actually a bound state of a five-brane with a number of membranes: the quantum numbers s_{mn} count the number of membranes that are wrapped around the 2-cycle T_{mn}^2 that is dual to T_{mn}^3 . This is similar to Witten's description of bound states of (p, q) -strings [15]. Notice that in this case that the zero-mode of the canonical momentum π_U is also quantized, but these are not independent because the self-duality condition implies that $\pi_U = dU$.

Our aim in this paper is to arrive at a U-duality invariant description of the five-brane. We have shown that under the U-duality group $SO(5, 5, \mathbf{Z})$ the charges s_{mn} are part of an irreducible 16-dimensional spinor representation together with the Kaluza-Klein momenta r^m of the five-brane and its winding number q around the five-torus. In fact, the spinor representation of $SO(5, 5)$ is naturally identified with the odd (co)homology of the five-torus T^5 . This observation motivates us to try write the other charges q and r^m as fluxes of 5- and 1-form field strengths through the 5-cycle and 1-cycles on T^5 . Indeed, the winding number of the five-brane around the internal T^5 can be written as

$$q = \int_{T^5} dV, \quad (3.2)$$

where dV is interpreted as the pull-back of the constant volume element on the T^5 -manifold. We now would like to turn the 4-form potential V into an independent field that is part of the world-brane theory.*

As mentioned, the effective action on the world-brane contains as bosonic fields, besides the tensor field U , five scalars. In the following we will interpret four of these as being the transversal coordinates in space-time. This leaves us with one additional scalar Y . In particular, since the world-brane is 6-dimensional, we can dualize it and obtain a four-form V . We will now identify this field with the field V in (3.2). Its five-form field strength $W = dV$ is normalized such that the flux q is integer, but for the rest it can take any value including zero. Now, formally we can go back to the description in terms of the dual scalar field Y by taking W to be an independent five-form and introducing Y as the Lagrange multiplier that imposes the Bianchi identity $dW = 0$. The fact that W has integral fluxes implies that Y must be a periodic field, *i.e.* $Y \equiv Y + r$ with r integer. Notice that on-shell $dY = \delta S / \delta \dot{V} = \pi_V$, and so dY is the canonical momentum for V . The 5 remaining charges r^m can now be identified with the integer winding numbers of Y around the 5 one-cycles T_m^1 :

$$r^m = \int_{T_m^1} dY. \quad (3.3)$$

It is straightforward (see also the previous footnote) to show that these operators are the generators of translations along the internal directions on the 5-torus, and so they indeed represent the Kaluza-Klein momenta of the five-brane. In this way all the 16 charges (q, r^m, s_{mn}) have been written as fluxes through the odd homology cycles on the five-brane, and so, in view of our previous remark, are naturally identified with a 16 component $SO(5, 5, \mathbf{Z})$ spinor.

To make this more manifest it is convenient to combine the fields Y , U and V , and their field strengths dY , dU and dV using $SO(5)$ gamma-matrices as

$$\begin{aligned} Y_{ab} &= Y \mathbf{1}_{ab} + *V_m \Gamma_{ab}^m + U_{mn} \Gamma_{ab}^{mn}, \\ (\nabla Y)_{ab} &= *dV \mathbf{1}_{ab} + (dY)_m \Gamma_{ab}^m + (*dU)_{mn} \Gamma_{ab}^{mn}, \end{aligned} \quad (3.4)$$

where $\nabla_{ab} = \Gamma_{ab}^m \partial_m$. The field Y_{ab} is not an unconstrained field, since it would describe too many degrees of freedom. Namely, we still have to impose the condition that dU is self-dual and dY is the dual of dV . This can be done in a rather elegant way in a

*In fact, in a light-cone formalism for extended objects one naturally obtains a residual gauge symmetry under volume preserving diffeomorphisms [20]. For our five-brane these can be used to eliminate all dependence on the five compact embedding coordinates X^m except the volume-form $dV = dX^m \wedge dX^n \wedge dX^k \wedge dX^l \wedge dX^p \epsilon_{mnpkl}$. Hence, the five spatial components $*V^m$ are basically the embedding coordinates X^m . Notice also that the gauge-transformation $V \rightarrow V + d\Lambda$ corresponds to a volume preserving diffeomorphism on the five-brane, since it leaves dV invariant.

canonical formalism be imposing[†] the constraint $\pi_Y^{ab} = \nabla Y^{ab}$, where π_Y^{ab} is the canonical conjugate momentum of Y_{ab} . This constraint reduces the number of on-shell degrees of freedom from 8 to 4.

The advantage of the notation (3.4) is that it makes the action of the U-duality group more manifest: the action of $SO(5) \times SO(5)$ is from the left and right respectively, while the other generators of $SO(5, 5)$ act as in (2.8). The results of this section can now be summarized by the statement that the central charge Z_{ab} coincides with the zero-mode part of $(\nabla Y)_{ab}$.

4. SPACE-TIME SUPERSYMMETRY

Our aim in this section is to investigate the BPS spectrum from the view point of the world-brane theory. At present we do not know a consistent quantum theory for five-branes that is derived from a covariant world-volume action. Fortunately, our only aim is to study the quantum states of the five-brane that are part of the space-time BPS spectrum, and, as we will see, for this we do not need to consider the full five-brane dynamics. Furthermore, even without using the details of the world-brane theory one can already say a lot about its quantum properties just on the basis of symmetry considerations and other general principles. Our only assumption is that the five-brane theory permits a light-cone gauge, so that there are 4 transversal coordinates X^i in the 6 uncompactified dimension. In the following sections this assumption will be justified by the fact that from this starting point we are able to derive a Lorentz invariant BPS spectrum.

In the light-cone gauge the $SO(5, 1)$ space-time Lorentz group is broken to the $SO(4)$ subgroup of transversal rotations. On the world-brane this group plays the role of an R-symmetry. We organize the fields accordingly using $SO(4)$ representations with $\alpha, \dot{\alpha}$ indicating the two chirality spinors. In addition our fields may carry one or two spinor indices of the $SO(5)$ group of spatial rotations on the five-brane. These are denoted by a, b , etc. In this notation we have the following fields on the five-brane

$$X^{\alpha\dot{\beta}}, \psi_b^\alpha, \psi_a^{\dot{\beta}}, Y_{ab}, \quad (4.1)$$

where Y_{ab} is the field that we introduced in (3.4). Notice that each of these fields has four on-shell components. These fields represent the collective modes of the five-brane and

[†]This generalizes the condition of self-duality to interacting theories, since we do not need to assume that the field is described by a free action.

their zero-modes will be used to construct the space-time supersymmetry algebra. More precisely, each of these fields has a canonical conjugate momentum $\pi_X^{\alpha\dot{\beta}}, \pi_\psi^{\alpha b}, \pi_\psi^{\dot{\beta} a}$ and π_Y^{ab} . Their zero-modes

$$p^{\alpha\dot{\beta}}, S_b^\alpha, S_a^{\dot{\beta}}, Z_{ab} \quad (4.2)$$

enter in the $N = (4, 4)$ space-time supersymmetry algebra respectively as the transversal momentum, part of the space-time supercharges (namely those that are broken by the five-brane) and the central charge.

The world-brane theory carries a chiral $N = (4, 0)$ supersymmetry that is generated by a set of supercharges G_a^α and $G_a^{\dot{\alpha}}$. These supercharges satisfy in general the commutator algebra

$$\begin{aligned} \{G_a^\alpha, G_b^\beta\} &= 2\epsilon^{\alpha\beta}(\mathbf{1}_{ab}H + \Gamma_{ab}^m(P_m + W_m)), \\ \{G_a^{\dot{\alpha}}, G_b^{\dot{\beta}}\} &= 2\epsilon^{\dot{\alpha}\dot{\beta}}(\mathbf{1}_{ab}H + \Gamma_{ab}^m(P_m - W_m)). \end{aligned} \quad (4.3)$$

Here H is the Hamiltonian on the five-brane, and P_m are world-brane momentum operators that generate translations in the five spatial directions. It will sometimes be convenient to combine them into the matrix

$$P_{ab} = \mathbf{1}_{ab}H + \Gamma_{ab}^m P_m. \quad (4.4)$$

The operators P_m play the same role as $L_0 - \bar{L}_0$ in string theory, and as is clear from this analogy, will have to annihilate the physical states in the spectrum of the five-brane

$$P_m|\text{phys}\rangle = 0. \quad (4.5)$$

The operator W_m that appear in the supersymmetry algebra (4.3) represents a possible vector-like central charge. In terms of the fluxes of dY and $T = dU$ it receives a contribution given by the topological charge

$$W_m = \int_{T_m^4} dY \wedge T. \quad (4.6)$$

In order to have a realization of space-time supersymmetry (without space-time vector central charges) we will also have to put W_m to zero on physical states

$$W_m|\text{phys}\rangle = 0. \quad (4.7)$$

If we do not introduce extra degrees of freedom, this condition implies the relation $K_L = K_R$. Finally, from the light-cone condition $x^+ = p^+ \tau$, we find that we have to impose the condition

$$(H - p^+ p^-)|\text{phys}\rangle = 0. \quad (4.8)$$

We will now turn to a discussion of the space-time supersymmetry. To reduce the number of equations somewhat, we will concentrate first on the “left-movers” G_a^α , and discuss the dotted “right-moving” components $G_a^{\dot{\alpha}}$ afterwards. The world-brane supercharges represent the unbroken part of the full $N = (4, 4)$ space-time supersymmetry algebra. The other generators must be identified with the zero-modes S_a^α and $S_a^{\dot{\alpha}}$ of the conjugate momenta of the world-brane fermions, since these are the Goldstone modes associated with the broken supersymmetry. Under the world-brane supersymmetry these zero-modes transform into the zero-modes of the bosonic fields,

$$\begin{aligned}\{G_a^\alpha, S_b^\beta\} &= \epsilon^{\alpha\beta} Z_{ab}, \\ \{G_a^\alpha, S_b^{\dot{\beta}}\} &= \omega_{ab} \not{p}^{\alpha\dot{\beta}}.\end{aligned}\tag{4.9}$$

To get the complete set of relations one needs to use the world-brane supersymmetry algebra (4.3) together with the conditions (4.8) and (4.5). This gives on physical states

$$\begin{aligned}\{G_a^\alpha, G_b^\beta\} &= 2p^+ p^- \epsilon^{\alpha\beta} \omega_{ab}, \\ \{S_a^\alpha, S_b^\beta\} &= \epsilon^{\alpha\beta} \omega_{ab},\end{aligned}\tag{4.10}$$

and similarly for the dotted components. The space-time supersymmetry generators are therefore

$$Q_\alpha^a = (\sqrt{2p^+} S_a^{\dot{\alpha}}, G_a^\alpha / \sqrt{p^+}),\tag{4.11}$$

where on the left-hand side α denotes a chiral four-component $SO(5, 1)$ space-time spinor index.

We can now discuss the space-time BPS states from the world-brane point of view. The value of the BPS mass is determined by the central charge Z_{ab} , and so we know that for a BPS state we should find that

$$H|\text{BPS}\rangle = \frac{1}{2}(p_i^2 + m_{\text{BPS}}^2)|\text{BPS}\rangle\tag{4.12}$$

where m_{BPS}^2 is given in (2.7). We will now prove that this is in fact the lowest eigenvalue of the Hamiltonian H in the sector with a given central charge Z_{ab} . For this purpose, let us introduce new operators \hat{G}_a^α and \hat{P}_{ab} which are defined as the non-zero mode contributions of the respective operators. They satisfy the relations

$$\begin{aligned}G_a^\alpha &= Z_a^{\dagger b} S_b^\alpha + \not{p}^{\alpha\dot{\beta}} S_{\dot{\beta}, a} + \hat{G}_a^\alpha, \\ P_{ab} &= \frac{1}{2} p_i^2 \delta_{ab} + \frac{1}{2} (Z^\dagger Z)_{ab} + \hat{P}_{ab}.\end{aligned}\tag{4.13}$$

Using the world-brane and space-time supersymmetry algebra we derive that the anti-commutator of two of these operators is

$$(\hat{G}^\dagger \hat{G})_{ab} \equiv \frac{1}{2} \epsilon_{\alpha\beta} \{\hat{G}_a^\alpha, \hat{G}_b^\beta\} = 2P_{ab} - p_i^2 \delta_{ab} - (Z^\dagger Z)_{ab}.\tag{4.14}$$

Next, by taking the trace with the non-negative definite matrix $\rho = \frac{1}{2}(1 + \hat{K} \cdot \Gamma)$, where \hat{K} is a unit vector in the direction of $K = K_L = K_R$, one deduces that

$$H + \hat{K} \cdot P = \frac{1}{2}p_i^2 + \frac{1}{2}m_{\text{BPS}}^2 + \frac{1}{2}\text{tr}(\rho \hat{G}^\dagger \hat{G}). \quad (4.15)$$

The last operator on the right-hand-side is clearly positive definite. Furthermore, since BPS states are physical, they have to be annihilated by the translation generators P^m . Combining these two facts gives the statement we wanted to prove. It also tells us that BPS states are annihilated by half of the operators \hat{G}_a

$$\varepsilon^a \hat{G}_a |\text{BPS}\rangle = 0. \quad (4.16)$$

In this sense, space-time BPS states are also BPS states from the world-brane point of view.

5. STRINGS ON THE FIVE-BRANE

The U-duality group acts on the shape and size of the internal 5 torus, and in particular contains transformations that map large to small volumes. If we require that the five-brane theory is invariant under such transformations, it is clearly necessary to include short distance degrees of freedom. Furthermore, these extra degrees of freedom need to behave the same for very small box sizes as momentum modes do for large box sizes. This suggests that we can possibly restore the full U-duality invariance by replacing the world-brane theory by a string theory. Another independent indication that the five-brane world-volume theory may in fact contain string-like excitations is the possible occurrence of vector-like central charges in the supersymmetry algebra, since only one-dimensional extended objects can carry such charges.

One easily sees that BPS states necessarily correspond to the string ground states. These states are annihilated by 1/4 of the *space-time* supercharges, which implies, as we have seen in the previous section, that they are annihilated by 1/2 of *five-brane* supercharges. If we repeat this procedure once more, we conclude that in terms of the string, BPS states are annihilated by all of the *string* supercharges. Although we only use the string ground states in this paper (see however [17]) we will now make some remarks concerning the formulation of this six-dimensional string theory.

We are looking for a string model whose ground states represent the massless tensor multiplet describing the collective modes of the five-brane. Specifically, we expect ground

states of the form $|\alpha\dot{\beta}\rangle$ that describe the four scalars $X^{\alpha\dot{\beta}} = \sigma_i^{\alpha\dot{\beta}} X^i$, states $|\alpha b\rangle$ and $|a\dot{\beta}\rangle$ that describe the 4 world-brane fermions ψ^α and $\psi^{\dot{\alpha}}$, and finally we need states $|ab\rangle$ that correspond to the fifth scalar Y and the 3 helicity states of the tensor field U . Here the indices $a, b = 1, 2$ now label chiral $SO(4)$ spinors.*

This structure arises naturally in the following model. We will assume that the world-sheet theory of this string theory can be formulated in a light-cone gauge, and so one expects to have 4 transversal bosonic coordinate fields $x^{a\dot{a}}(z, \bar{z})$ together with fermionic partners $\lambda_\alpha^a(z)$, $\bar{\lambda}_{\dot{a}}^a(\bar{z})$. The world-sheet theory has 4 left-moving and 4 right-moving supercharges $F^{\dot{a}\alpha} = \oint \partial x^{\dot{a}b} \lambda_b^\alpha$ and $\bar{F}^{\dot{a}\dot{\alpha}} = \oint \bar{\partial} x^{\dot{a}b} \bar{\lambda}_b^{\dot{\alpha}}$ which correspond to the unbroken part of the world-brane supersymmetry and satisfy

$$\{F^{\dot{a}\alpha}, F^{\dot{b}\beta}\} = 2\epsilon^{\dot{a}\dot{b}}\epsilon^{\alpha\beta} L_0, \quad (5.1)$$

where L_0 is the left-moving world-sheet Hamiltonian. The ground states must form a multiplet of the zero-mode algebra $\{\lambda_\alpha^a, \lambda_b^\beta\} = \epsilon_{ab}\epsilon^{\alpha\beta}$. This gives 2 left-moving bosonic ground states $|\alpha\rangle$ and 2 fermionic states $|a\rangle$. By taking the tensor product with the right-moving vacua one obtains in total 16 ground states

$$\left(|\alpha, k_L\rangle \oplus |a, k_L\rangle\right) \otimes \left(|\dot{\beta}, k_R\rangle \oplus |b, k_R\rangle\right), \quad (5.2)$$

just as we wanted. Here we also took into account the momenta (k_L, k_R) , which form a $\Gamma_{5,5}$ lattice, since we have assumed that the five-brane has the topology of T^5 . Since the light-cone coordinates parametrize a cylinder $S^1 \times \mathbf{R}$, the standard light-cone formalism has to be slightly modified, as we have to take into account that the string can also wind around this circle. Specifically, we must impose the mass-shell condition

$$L_0 = k_L^+ k_L^- = \frac{1}{2} k_0^2 - \frac{1}{2} (k_L^5)^2, \quad (5.3)$$

so that the level matching condition between the left-moving and right-moving sectors of the string becomes

$$L_0 - \bar{L}_0 = \frac{1}{2} (k_L^5)^2 - \frac{1}{2} (k_R^5)^2 = k^5 w^5. \quad (5.4)$$

with w^5 the winding number around the S^1 . Level matching implies that for the ground states $|k_L| = |k_R|$.

*Though the string theory we want needs to contain the tensor field U in its spectrum, note that it should not carry any charge with respect to it, since this would violate charge conservation on the five-brane. Rather, the U -field should couple via its field strength T , like an RR-vertex operator. The string described in this section should therefore be distinguished from the self-dual string considered in [21]. We thank M. Becker, J. Polchinski, and A. Strominger for discussions on this point.

The world-sheet supersymmetry transformation relates the fermion zero-modes and the transversal momentum k_L of the string

$$\{F^{\dot{a}\alpha}, \lambda^{b\beta}\} = k_L^{ab} \epsilon^{\alpha\beta}. \quad (5.5)$$

Hence one can construct a (4,0) supersymmetry on the 5+1-dimensional target space of the string, *i.e.* on the world-volume of the five-brane, as follows

$$\hat{G}^{a\alpha} = (\sqrt{2k_L^+} \lambda^{a\alpha}, F^{\dot{a}\alpha} / \sqrt{k_L^+}), \quad (5.6)$$

where on the left-hand side a denotes a chiral SO(5,1) spinor index. These generate the algebra

$$\{\hat{G}^{a\alpha}, \hat{G}^{b\beta}\} = 2\epsilon^{\alpha\beta} (k^0 \mathbf{1}^{ab} + k_L^m \Gamma_m^{ab}) \quad (5.7)$$

and similar for the right-moving generators.

We now propose that the world-brane dynamics of a single five-brane is described by the second quantization of this string theory. The most important consequence for our study of the BPS states is that the world-brane supersymmetry algebra gets modified. Namely, as we have just shown, the anti-commutator of the left-moving supercharges produces the left-moving momentum k_L , while the right-movers give k_R . Hence the string states form representations of the $N = (4, 0)$ supersymmetry algebra[†]

$$\begin{aligned} \{G^{a\alpha}, G^{b\beta}\} &= 2\epsilon^{\alpha\beta} (H \mathbf{1}^{ab} + P_L^m \Gamma_m^{ab}), \\ \{G^{a\dot{\alpha}}, G^{b\dot{\beta}}\} &= 2\epsilon^{\dot{\alpha}\dot{\beta}} (H \mathbf{1}^{ab} + P_R^m \Gamma_m^{ab}). \end{aligned} \quad (5.8)$$

The operators H , P_L^m and P_R^m act on multi-string states that form the Hilbert space of the five-brane. The five-brane Hamiltonian H measures the energy of the collection of strings, while $\frac{1}{2}(P_L^m + P_R^m)$ measure the total momentum. But we see that the algebra also naturally contains a vector central charge $W^M = \frac{1}{2}(P_L^m - P_R^m)$, which measures the sum of the string winding numbers around the 5 independent one-cycles on the world-brane.

We are interested in BPS states, which are annihilated by eight of the world-brane supersymmetry generators. Such states are obtained by combining individual string states $|\text{bps}\rangle$ that satisfy

$$\varepsilon_{a\alpha} \hat{G}^{a\alpha} |\text{bps}\rangle = 0, \quad (5.9)$$

where ε is constrained by

$$k_L^{ab} \varepsilon_{b\alpha} = 0, \quad (5.10)$$

[†]As explained in section 4, the appropriate second-quantized generators $G^{a\alpha}$ also contain a zero-mode contribution in addition to the part $\hat{G}^{a\alpha}$ which is expressed in terms of the string creation and annihilation operators. We have silently added the zero-mode contribution here.

and similarly for the opposite chirality. In terms of the world-sheet generators this condition reads

$$F^{a\alpha}|\text{bps}\rangle = k_L^{ab}\lambda_{b\alpha}|\text{bps}\rangle. \quad (5.11)$$

This condition tells us that the string must be in the left-moving ground state. The full BPS condition thus implies that the individual strings must be in their ground state.

6. U-DUALITY INVARIANT BPS SPECTRUM OF THE FIVE-BRANE

We will now discuss the BPS quantization of the five-brane. A natural starting point is the low-energy effective field theory on the world-volume, which is appropriate for describing the world-brane dynamics at large volume. Our hypothesis is that the effective theory in this regime takes the most simple form consistent with all the symmetries of the five-brane. As we have argued, this minimal requirement is fulfilled by the field theory consisting of the single tensor multiplet described in sections 3 and 4. These fields describe the ground states of the strings, except that at small volumes we have to take into account the winding configurations of the underlying string theory. We will further make the assumption that for the purpose of constructing the BPS spectrum it is allowed to treat the five-brane fluctuations in a linear approximation, that is, we will use free field theory.

Since we will work towards a Hamiltonian formalism, we will from the beginning distinguish the time-coordinate τ from the five space-like coordinates σ^m . We can further pick a Coulomb gauge for the two-form field U by demanding $U_{m0} = 0$ and $\partial^m U_{mn} = 0$. It is useful to introduce the matrix-valued derivative

$$D_{ab} = \mathbf{1}_{ab}\partial_0 + \Gamma_{ab}^m\partial_m \quad (6.1)$$

and the world-volume bi-spinor field Y^{ab} introduced in equation (3.4). Its equation of motion then takes the form

$$D_a{}^b(DY)_{bc} = D_a{}^b(DY)_{bc}^\dagger = 0, \quad (6.2)$$

where D acts on Y via matrix multiplication. These two equations reduce the number of independent on-shell components of Y_{ab} to four. The free field equations of motion of the other fields are in this notation

$$\begin{aligned} D^{ab}(DX)_{ab}^{\alpha\dot{\beta}} &= 0, \\ (D\psi)_a^\alpha &= 0. \end{aligned} \quad (6.3)$$

The free field Hamiltonian and momentum operators on the world-brane take the quadratic form

$$P^{ab} = \int_{T^5} d^5\sigma \frac{1}{2} \left[(DY)^{ac} (DY)_c^{\dagger b} + (DX_i)^{ac} (DX_i)_c^b + \psi_\alpha^a (D\psi)^{b\alpha} + \psi_{\dot{\alpha}}^a (D\psi)^{b\dot{\alpha}} \right] \quad (6.4)$$

To describe the quantum fluctuations of the five-brane, we expand the various fields in plane waves that propagate on the world-volume. Since we assume that the world-volume has the topology of $T^5 \times \mathbf{R}$, we can label these waves with 5 integral momenta k_m . For example, the expansion of Y_{ab} is

$$(DY)_{ab} = Z_{ab} + \sum_k (\mathbf{a}_I^\dagger(k) u_{ab}^I(k) e^{ik \cdot \sigma} + \mathbf{a}_I(k) \bar{u}_{ab}^I(k) e^{-ik \cdot \sigma}), \quad (6.5)$$

where I runs from 1 to 4, and k runs over the five-dimensional momentum lattice of T^5 . Together with the four other bosonic fields $X^{\alpha\dot{\beta}}$ and fermionic fields ψ_a^α and $\psi_a^{\dot{\alpha}}$, we obtain creation and annihilation modes $\mathbf{a}_I(k)$, $\psi_A(k)$, where the indices I and A now both run from 1 to 8.

We could now impose the BPS conditions in the light-cone formalism that we have been using as described in section 4. However, as we have seen, the effective field theory approach is only sufficient (and consistent) in the special case that the central charge satisfies the condition $W^m = \frac{1}{2}(K_L^m - K_R^m) = 0$. This condition, which breaks U-duality, can be understood as the absence of vector central charges on the world-brane. In terms of the string theory this corresponds to considering the sector with zero total winding number. Instead of working out this case in detail, we will immediately give a manifestly U-duality invariant derivation of the BPS spectrum. The field theory limit can simply be obtained afterwards by simply putting W^m to zero.

Indeed, it is clear from the discussion in section 5 that a field theory description is incomplete and that at small volumes of T^5 extra degrees of freedom must be included in the effective description. In particular, the presence of string winding states will imply that the modes of the quantum fields can carry besides the momentum k also a winding number w . To put this idea into effect, we note that the equation of motion (6.2) of the field DY in momentum space can indeed be generalized to an $SO(5, 5)$ invariant equation by introducing a left- and a right-momentum vector, as follows

$$\begin{aligned} (|k| \mathbf{1} - k_L \cdot \Gamma)^{ab} u_{bc}^I(k_L, k_R) &= 0, \\ (|k| \mathbf{1} - k_R \cdot \Gamma)^{ab} u_{bc}^{\dagger I}(k_L, k_R) &= 0. \end{aligned} \quad (6.6)$$

Here $u_{ab}^I(k_L, k_R)$ denotes the generalization of the bi-spinors $u_{ab}^I(k)$ in the mode expansion (6.5) of DY_{ab} . Via the action of $SO(5, 5)$ on the spinor indices of DY , we deduce that the

pair of momenta k_L and k_R combine into an $SO(5, 5)$ null-vector, with $k_L^2 - k_R^2 = 0$. In a similar way, we can argue that all other fields must also depend on two momenta instead of one.

We are thus led to consider a Fock space with stringy oscillators $\mathbf{a}^I(k_L, k_R), \psi^A(k_L, k_R)$ where $(k_L, k_R) \in \Gamma^{5,5}$ and $|k_L| = |k_R| = |k|$. The generalized number operator is

$$N_{k_L, k_R} = \mathbf{a}_I^\dagger(k_L, k_R) \mathbf{a}^I(k_L, k_R) + \psi_A^\dagger(k_L, k_R) \psi^A(k_L, k_R). \quad (6.7)$$

We can now write the Hamiltonian and momentum operators on the five-brane (see equation (6.4)) in terms of the contributions of the zero-modes and these particular string modes as

$$\begin{aligned} H &= \frac{1}{2}m_0^2 + \frac{1}{2}p_i^2 + \sum_{k_L, k_R} |k| N_{k_L, k_R}, \\ P_L^m &= K_L^m + \sum_{k_L, k_R} k_L^m N_{k_L, k_R}. \end{aligned} \quad (6.8)$$

where $m_0^2 = \frac{1}{4}\text{tr} Z^\dagger Z$ and $K_L^m = \frac{1}{8}\text{tr}(\Gamma^m Z^\dagger Z)$. Similarly, we define P_R^m with $K_R^m = \frac{1}{8}\text{tr}(\Gamma^m Z Z^\dagger)$.

Note that we now have independent left-moving and right-moving momentum operators P_L and P_R on the five-brane. In order to be able to realize the space-time supersymmetry algebra, we have to impose on physical states the conditions that both momentum operators vanish

$$P_L^m |\text{phys}\rangle = P_R^m |\text{phys}\rangle = 0. \quad (6.9)$$

These equations tell us that the sum of the individual left-moving or right-moving string momenta k_L, k_R have to cancel the contribution K_L, K_R of the zero-mode fluxes on the five-brane. We see in particular that, in the case that $K_L \neq K_R$, this necessitates the inclusion of string winding modes.

We will now impose the BPS condition. Hereto we can use the result demonstrated at the end of section 4, that BPS states saturate the lower bound (4.15) for the “light-cone Hamiltonian” $H + \hat{K} \cdot P_{L,R}$ for a given value of the central charge Z_{ab} . In more detail we proceed as follows. The (mass)² of these five-brane states is measured by the operator

$$m^2 = m_0^2 + \sum_{k_L, k_R} 2|k| N_{k_L, k_R}. \quad (6.10)$$

As we have explained in detail in section 2, the BPS condition implies that m^2 must be equal to $m_{\text{BPS}}^2 = m_0^2 + 2|K_{L,R}|$ which is the highest eigenvalue of $Z^\dagger Z$ or $Z Z^\dagger$. Using the physical state condition $P_L = 0$, the BPS mass formula may be rewritten as

$$\frac{1}{2}m^2 + \hat{K}_L \cdot P_L = \frac{1}{2}m_0^2 + |K_L|, \quad (6.11)$$

where we introduced the unit vector \hat{K}_L in the direction of K_L . Inserting the mode-expansions of P_L and m^2 shows that the zero-mode part of the left-hand side is already equal to the right-hand-side. The remaining oscillator part must therefore add up to zero

$$\sum_{k_L, k_R} (|k| + \hat{K}_L \cdot k_L) N_{k_L, k_R} = 0. \quad (6.12)$$

The left-hand side is a non-negative expression and can vanish only if the excitations have momentum k_L directed in the opposite direction of K_L . Similarly, one obtains an analogous result for the right-moving momenta k_R . So, we conclude that the BPS spectrum for given central charge vector $K = (K_L, K_R) \in \Gamma^{5,5}$ is obtained by acting with only those oscillators \mathbf{a}_k and ψ_k for which the momentum vector $k = (k_L, k_R)$ points in the direction opposite to K .

To count the number of BPS states for given vector $K \in \Gamma^{5,5}$, let $[K]$ be the largest positive integer so that $K/[K]$ is still an integral vector. In other words, K is $[K]$ times a primitive vector, and the $[K]$ can be defined as the greatest common divisor of the ten integers K_L^m, K_R^m . With this notation, the allowed momenta of the oscillators must be of the form $k_n = -nK/[K]$, for some positive integer n . Let N_n denote the occupation number of these modes. The level matching conditions (6.9) together with (6.8) now reduce to the simple combinatorial relation

$$\sum_n n N_n = [K]. \quad (6.13)$$

Since there are 8 bosonic and 8 fermionic modes that contribute at each oscillator level, we obtain the result that we announced at the end of section 2. The number of BPS states is given by $d([K])$ with

$$\sum d(N) t^N = (16)^2 \prod_n \left(\frac{1+t^n}{1-t^n} \right)^8. \quad (6.14)$$

We would like to point out that the above result has a rather striking interpretation. We see that in the BPS limit the excitations of the five-brane are constrained to lie in a single space-like direction, which is determined by the value of the central charge. So, effectively the six-dimensional world-volume reduces to a world-sheet and the BPS five-brane behaves like a string, in fact a chiral type II string. The U-duality group $SO(5, 5, \mathbf{Z})$ acts on the momentum vector K and so permutes the various string-like excitations of the five-brane.

If we want to relate this eleven-dimensional point of view to the ten-dimensional type II string, we have to single out a particular direction on the world-brane. As we discussed

before, this breaks the U-duality group to the little group $SO(4, 4, \mathbf{Z})$ and corresponds to distinguishing NS-NS and R-R type charges. The string-like excitations in this fixed direction then correspond directly to the BPS states of the fundamental type II string.

COMPARISON WITH D-BRANES

It might be illustrative to compare the counting of BPS states in this paper with the more conventional counting using D-branes [12]. It is especially instructive to see how the string degrees of freedom of the five-brane, which from our point of view were a crucial ingredient in a complete description of the BPS spectrum, manifest themselves in the world-volume theories of the D-branes. In particular we would like to see to which extent string-like excitations occur for the Dirichlet four-brane in type IIA superstring theory, which can be considered as a simultaneous dimensional reduction to ten dimensions of the five-brane in M-theory.

So let us reconsider the compactification of the type IIA superstring on a four-torus. As we mentioned in section 2, the 16 quantum numbers now decompose in the 8 NS-NS and 8 R-R charges

$$\begin{aligned} (p, w) &\in H^{odd}(T^4), \\ (q_0, q_2, q_4) &\in H^{even}(T^4) \end{aligned} \tag{6.15}$$

Here p_i, w^i are the momenta and winding number of the fundamental NS string, and q_p ($p = 0, 2, 4$) is the Dirichlet p -brane charge. In terms of these charges the 10 components of the vector (K^m, W_m) of equation (2.11) are given by

$$\begin{aligned} K^5 &= q_4 q_0 + \frac{1}{2} q_2 \wedge q_2 \\ K^i &= q_4 p^i + (q_2 \wedge w)^i \\ W_5 &= p^i w_i \\ W_i &= q_0 w_i + (q_2 \cdot p)_i \end{aligned} \tag{6.16}$$

Our analysis and U-duality predict that the degeneracy is a function of the g.c.d. of these 10 integers.

Let us now look at configurations in which we can make a meaningful comparison with standard D-brane computations to count bound state degeneracies. We will be particularly interested in configurations with a single 4-brane, no 2-brane (for simplicity) and an arbitrary number of 0-branes, *i.e.* we put $q_4 = 1$ and $q_2 = 0$. In that case the degeneracy is given by $d(N)$ with

$$N = gcd(q_0, p^i, p \cdot w, q_0 w^i) = gcd(q_0, p^i). \tag{6.17}$$

Can we make a macroscopic BPS string-state on the four-brane which has a non-zero winding number W^i ? Such an object will manifest itself, in the large volume, as a long string-like object. In our philosophy such a state is a coherent sum of BPS five-brane strings with parallel momenta and winding numbers.

If we choose for simplicity the total momentum K^i to be zero, we see from the above expressions that it is indeed quite easy to make such an object. For $q_0 = 1$ and $p^i = 0$ the winding number W^i simply equals the winding number w^i of the fundamental NS string. Furthermore, the degeneracy of such a state is given by the usual number of string ground states $d(0) = 2^8$. So in this case, the string BPS state is nothing but the fundamental closed string bound with a zero-brane to the four-brane.

In fact, this point of view can be easily generalized to a configuration with an arbitrary number q_0 of zero-branes, where the degeneracy is predicted to be $d(q_0)$. There is a simple explanation of this counting using the so-called “necklace” model. In this case we have q_0 zero-brane “beads” that are stringed together with a NS string and bound to the four-brane. These zero-branes will cut the string in q_0 pieces, each of which has the usual $8+8$ ground states. However, the zero-branes can cluster together and form bound states. Therefore, to compute the total number of BPS states we have to sum over partitions giving the usual degeneracy formula

$$\sum d(N)t^N = 2^8 \prod_n \left(\frac{1+t^n}{1-t^n} \right)^8 \quad (6.18)$$

7. HETEROTIC STRING THEORY FROM THE FIVE-BRANE.

The above results can be generalized in a straightforward fashion to other internal manifolds than T^5 . As an explicit example, we will now briefly discuss the cases of $S^1 \times K3$ and $T^4 \times S^1/\mathbf{Z}_2$. Previous studies of M-theory compactified on these manifolds [6, 7, 22] have shown that in both cases the theory becomes equivalent to the heterotic string compactified on T^4 (provided that the \mathbf{Z}_2 action in the latter case is defined appropriately). Here we confirm this by means of an explicit study of the five-brane BPS spectrum. In particular we will find that the two different compactifications are T-dual from the point of view of the world-volume string theory.

It will be useful in the following to think about $K3$ as (a resolution of) the orbifold T^4/\mathbf{Z}_2 , so that both types of compactification manifolds are obtained as \mathbf{Z}_2 -orbifolds of T^5 , with the two \mathbf{Z}_2 ’s acting on T^4 and S^1 respectively. These two transformations are

represented on the fluxes Z^{ab} of DY^{ab} as follows

$$Z \rightarrow \Gamma_5 Z \Gamma_5 \quad (7.1)$$

and

$$Z \rightarrow -\Gamma_5 Z \Gamma_5 \quad (7.2)$$

respectively. To be able to construct the appropriate orbifolds we will first have to extend these \mathbf{Z}_2 -actions to a transformation on the complete fields, and furthermore, check that the resulting transformations are indeed symmetries of the five-brane theory. Notice that under both \mathbf{Z}_2 actions the quantities K^i and W_i (with $i = 1, \dots, 4$) are odd, while K^5 and W_5 are invariant. This suggest that the \mathbf{Z}_2 action must be accompanied by a reflection of the world-volume of the five-brane. More precisely, we find that the unique \mathbf{Z}_2 -symmetry of the world-volume action that incorporates the above transformation on the fluxes is given by the orientifold transformation

$$\begin{aligned} DY(\sigma) &\rightarrow \pm \Gamma_5 \tilde{D}Y(\tilde{\sigma}) \Gamma_5, \\ DX(\sigma) &\rightarrow DX(\tilde{\sigma}), \\ \psi(\sigma) &\rightarrow \Gamma_5 \psi(\tilde{\sigma}), \end{aligned} \quad (7.3)$$

where $\tilde{\sigma}^i = -\sigma^i$ for $i = 1, \dots, 4$, and $\tilde{\sigma}^5 = \sigma^5$. Without this coordinate reflection this transformation would *not* be a symmetry. For example, it would not leave the definition of the translation operators P_m invariant.

So for both types of orbifolds, the world-volume coordinates σ lie on the same orbifold $S^1 \times T^4/\mathbf{Z}_2$, which we may also think of as $S^1 \times K3$. The two theories differ, however, via their boundary condition on the field Y . In both cases, the world-volume theory has a chiral $N = 2$ global supersymmetry and the R-symmetry group is reduced to $SU(2)$, which commutes with the holonomy group $SU(2)$ of $K3$. The 6-dimensional space-time theory has an unbroken $N = 2$ supersymmetry with central charge $(a, b = 1, 2)$

$$\{Q^a, \bar{Q}^b\} = Z^{ab}. \quad (7.4)$$

As before, this symmetry is realized on the five-brane via the world-volume supersymmetry generators G^a and fermion zero-modes S^a .

It is now straightforward to see that the two types of theories are related via the U -duality (or T -duality) transformation that interchanges the momentum and winding modes of the string theory on the five-brane. To see this we have to treat the two orbifolds separately. For the $K3 \times S^1$ compactification, of the fluxes introduced above for T^5 , only

the 8 \mathbf{Z}_2 -invariant fluxes q , r_5 and s_{ij} with $i, j = 1, \dots, 4$ survive. In addition, there are 16 extra fluxes

$$s_I = \int_{S^1 \times S_I^2} dU, \quad (7.5)$$

where the S_I^2 are the two-spheres surrounding the 16 fixed points on T^4/\mathbf{Z}_2 . The intersection form on the total set of fluxes has signature $(4, 20)$, and the integers therefore label a vector* (p_L, p_R) in the lattice $\Gamma^{4, 20}$. In terms of the quadratic quantities K_m, W_m the only non-vanishing component is given by

$$K_5 = \frac{1}{2}(p_L^2 - p_R^2). \quad (7.6)$$

In the dual compactification on $T^4 \times S^1/\mathbf{Z}_2$, on the other hand, we are left with the fluxes r_i and s_{i5} . These are complemented by 16 extra fluxes \tilde{s}_I , which again combine with the other fluxes into a vector $(\tilde{p}_L, \tilde{p}_R) \in \Gamma^{4, 20}$. In this case we find only a non-zero contribution to

$$W_5 = \frac{1}{2}(\tilde{p}_L^2 - \tilde{p}_R^2). \quad (7.7)$$

As we will now show, these results imply that after imposing the level matching constraints $P_L^5 = P_R^5 = 0$ and the BPS condition, only strings on the five-brane with either pure momentum or pure winding number in the 5-direction will contribute in BPS states. In particular, the T-duality map on this S^1 will interchange the momentum and winding modes and thus the two compactifications.

In both cases, the dependence of the central charge Z on the integer fluxes is parametrized by means of the action of $SO(4, 20)$ on the lattice $\Gamma^{4, 20}$. The central charge only depends on the 4 ‘left-moving’ components p_L (or \tilde{p}_L) via

$$Z^{ab} = p_L^i \sigma_i^{ab}. \quad (7.8)$$

BPS states again satisfy a condition of the form $(\varepsilon_a Q^a + \bar{\varepsilon}_a \overline{Q}^a)|\text{BPS}\rangle = 0$. In this case the central charge matrix Z satisfies $Z^\dagger Z = p_L^2 \mathbf{1}$, and the eigenvalue equation $Z^\dagger Z \varepsilon = m_{\text{BPS}}^2 \varepsilon$ has therefore two independent solutions, with eigenvalue

$$m_{\text{BPS}} = |p_L|. \quad (7.9)$$

The resulting BPS multiplets are 16 dimensional, which equals the dimension of the massless representations of the $N = (2, 2)$ space-time supersymmetry.

*This ‘left-right’ suffix will turn out to correspond to the two chiral sectors of the space-time heterotic string, and should *not* be confused with the label distinguishing left or right-moving string modes on the five-brane.

To obtain the multiplicities of the BPS states it is again necessary to introduce a mode expansion of the low-energy string fields on the world-volume. The modes are labeled by integral momentum $k = k^5$ and winding number $w = w^5$ in the S^1 direction together with a quantum number that labels the eigenmodes with eigenvalue h of the Laplacian on $K3$. The Hamiltonian takes the form

$$H = \frac{1}{2}p_i^2 + \frac{1}{2}p_L^2 + \frac{1}{2}p_R^2 + \sum_{k,w,h} \sqrt{k^2 + w^2 + h} N_{k,w,h} \quad (7.10)$$

Of the translation operators only the component P^5 in the direction of the S^1 survives. For the $S^1 \times K3$ compactification it takes the form

$$P^5 = \frac{1}{2}p_L^2 - \frac{1}{2}p_R^2 + \sum_{k,w,h} k N_{k,w,h}. \quad (7.11)$$

In a similar way as before, we deduce from combining the BPS mass-shell condition $H = \frac{1}{2}p_i^2 + \frac{1}{2}m_{\text{BPS}}^2$ with the constraint $P^5 = 0$, that charged BPS states can only oscillate in the left-moving S^1 direction, *i.e.* opposite to K^5 , while the winding number must vanish $w = 0$. The total number of oscillators is constrained by the level matching condition

$$\frac{1}{2}p_L^2 - \frac{1}{2}p_R^2 + \sum_k k N_{k,0,0} = 0. \quad (7.12)$$

In the dual compactification $T^4 \times S^1/\mathbf{Z}_2$ the only modification is the interchange of the momentum label k and the winding label w . The total winding number of the string is related to the fluxes via the constraint $W^5 = 0$, which reduces for BPS states to

$$\frac{1}{2}\tilde{p}_L^2 - \frac{1}{2}\tilde{p}_R^2 + \sum_w w N_{0,w,0} = 0. \quad (7.13)$$

This gives a concrete description of the BPS states in terms of the underlying string modes.

The number of oscillators that contribute to these expressions is determined by the number of harmonic zero-modes with $h = 0$ of the various fields on the $K3$ manifold. There are 5 bosonic oscillators \mathbf{a}_k^i with $i = 1, \dots, 5$ corresponding to the constant zero-modes of the coordinate fields X^i and Y on $K3$. The 19 anti-self-dual harmonic 2-forms on $K3$ lead to 19 additional left-moving modes \mathbf{a}_k^i with $i = 6, \dots, 24$. So in the end we are left with 24 left-moving bosonic oscillator modes \mathbf{a}_k^i , which can be recognized as the left-moving sector of the heterotic string compactified on T^4 to 6 dimensions.[†] Since also

[†]The 3 self-dual two-forms on $K3$ and the chiral covariantly constant spinor on $K3$ give rise to oscillator modes which are right-moving on S^1 , and form, together with the right-moving X^i and Y modes, the chiral world-sheet content of the superstring. These modes are however all eliminated via the BPS restriction.

the fluxes combine into a vector (p_L, p_R) on $\Gamma^{4,20}$, from this point on the counting of BPS states exactly parallels that of the heterotic string.

8. CONCLUDING REMARKS

In this paper we have presented a detailed analysis of the spectrum of BPS states of the five-brane theory in eleven dimensions. We motivated our proposed quantization procedure by using hints obtained from known results about BPS states in six-dimensional string theory, such as the symmetry under U-duality. Although our formulation was not manifestly Lorentz invariant in six dimensions, the final result for the spectrum of BPS states is in fact invariant under the full Lorentz group $SO(5,1)$. This is a consequence of the fact that the BPS restriction effectively reduces the five-brane dynamics to that of critical type II superstring theory. Hence we can use the standard derivation to show that our result for the complete BPS spectrum is indeed fully covariant.

Our derivation should be compared with the analysis of D-brane states [12]. In principle, it should be possible to make a concrete identification between specific five-brane excitations and configurations of D-branes and fundamental strings. Our construction of the BPS spectrum in terms of a Fock space in section 6 indeed matches the description of multiple D-brane configurations as given in [13]. Via this correspondence our results on the BPS spectrum also give useful information about the bound states of strings and D-branes [15].

We have concentrated on compactifications to 6 dimensions. But it is straightforward to extend our methods to compactifications to other dimensions. In particular, for dimensions higher than 6 one can deduce the BPS spectra by simply putting some of the 16 charges equal to zero. Thus, the results presented in this paper imply that at least for compactifications to six dimensions and higher, all BPS states can be obtained from the five-brane of M-theory. In particular, all the states that one would naively associate with the two-brane have now all become part of the spectrum of five-brane states.

This can be nicely illustrated in the specific example of the compactification on $T^4 \times S^1/\mathbf{Z}_2$ discussed in section 7 by considering the decompactification limit for large volume of the T^4 . Here one obtains the ten-dimensional heterotic string states as particular excitations of the five-brane. This should be compared to the observations made in [6] that from the eleven-dimensional point of view the heterotic string naturally arises from the two-brane wrapped around the S^1/\mathbf{Z}_2 . Our result however suggest that one should be able to think of the two-brane as a limiting configuration of the five-brane, in this case with the world-brane topology of $K3 \times S^1$. This could give an alternative explanation of the $E_8 \times E_8$ gauge symmetry.

To extend our formalism to dimensions five and lower, we have to consider BPS states that are annihilated by $1/8$ instead of $1/4$ of the space-time supercharges. This can be achieved by including vector central charges (both in space-time and on the world-volume) that modify the level matching conditions. In particular in five dimensions this extends the flux sectors to all 27 charges, that transform in the U-duality group $E_6(\mathbf{Z})$ [17]. Hence the five-brane also provides a unified description of all BPS states of 5 dimensional string theory. It would be interesting to see to which extent this approach can be used to obtain the complete string BPS spectrum in four dimensions.

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APPENDIX

Our conventions are the following. We denote the $SO(5)$ gamma-matrices as $\Gamma_m^a{}_b$ with $m = 1, \dots, 5$ and $a, b = 1, \dots, 5$. They satisfy

$$\{\Gamma_m, \Gamma_n\} = 2\delta_{mn}, \quad (\text{A.1})$$

and

$$\Gamma_m^\dagger = \Gamma_m, \quad \Gamma_m^T = \omega \Gamma_m \omega^T \quad (\text{A.2})$$

with $\omega = -\omega^T$ the symplectic form that gives the isomorphism $SO(5) \cong Sp(4)$. The other independent elements in the Clifford algebra are of the form

$$\Gamma_{mn} = \frac{1}{2}[\Gamma_m, \Gamma_n], \quad \Gamma_{mn}^\dagger = -\Gamma_{mn}. \quad (\text{A.3})$$

We can use the matrices $\mathbf{1}, \Gamma_m, \Gamma_{mn}$ to expand a general matrix Z that satisfies the reality condition

$$Z^* = \omega Z \omega^T \quad (\text{A.4})$$

as

$$Z = a\mathbf{1} + b^m \Gamma_m + c^{mn} \Gamma_{mn}. \quad (\text{A.5})$$

Note that

$$Z^\dagger = a\mathbf{1} + b^m \Gamma_m - c^{mn} \Gamma_{mn}, \quad (\text{A.6})$$

so, if in addition Z is hermitian, it is a linear combination of the identity and Γ_m .

In the text, we will use the form ω_{ab} to lower and raise indices of these matrices. Note that the matrices Γ_m^{ab} are now antisymmetric. In particular we will use the notation $\mathbf{1}_{ab} = \omega_{ab}$.

A matrix Z satisfying (A.4) forms a 16-dimensional Majorana-Weyl spinor representation of the group $SO(5, 5)$. In fact the $SO(5, 5)$ gamma matrices can be written as

$$\Gamma_m^L = \Gamma_m \otimes \mathbf{1} \otimes i\sigma_2, \quad (\text{A.7})$$

$$\Gamma_m^R = \mathbf{1} \otimes \Gamma_m \otimes \sigma_1, \quad (\text{A.8})$$

and satisfy

$$\{\Gamma_m^L, \Gamma_n^L\} = 2\delta_{mn}, \quad \{\Gamma_m^R, \Gamma_n^R\} = -2\delta_{mn}, \quad \{\Gamma_m^L, \Gamma_n^R\} = 0. \quad (\text{A.9})$$

In the chiral representation S^\pm with $\Gamma^{(11)} = \mathbf{1} \otimes \mathbf{1} \otimes \sigma_3 = \pm 1$, the generators of the $SO(5) \otimes SO(5)$ subgroup are given by $\Gamma_{mn} \otimes \mathbf{1}, \mathbf{1} \otimes \Gamma_{mn}$ whereas the off-diagonal generators are of the form $\pm \Gamma_m \otimes \Gamma_n$.

References

- [1] C. Hull and P. Townsend, “Unity of Superstring Dualities,” Nucl. Phys. **B 438** (1995) 109.
- [2] E. Witten, “String Theory in Various Dimensions,” Nucl. Phys. **B 443** (1995) 85.
- [3] M.J. Duff, P. Howe, T. Inami and K. S. Stelle, ”Superstrings in D=10 from supermembranes in D=11”, Phys. Lett. **191B** (1987) 70. M.J. Duff and J. X. Lu, ”Duality rotations in membrane theory”, Nucl. Phys. **B347** (1990) 394. M.J. Duff, J. T. Liu, and R. Minasian, ”Eleven-dimensional origin of string/string duality: a one-loop test”, Nucl. Phys. **B452** (1995) 261.
- [4] P. Townsend, “P-Brane Democracy,” hep-th/9507048.
- [5] J. Schwarz, “The Power of M-theory,” Phys. Lett. **367B** (1996) 97, hep-th/9510086; “M-theory Extensions of T-Duality,” hep-th/9601077.
- [6] P. Horava and E. Witten, “Heterotic and Type I String Dynamics from 11 Dimensions,” hep-th/9510209.
- [7] K. Dasgupta and S. Mukhi, “Orbifolds of M-Theory,” hep-th/9512196; E. Witten, “Five-branes and M-theory on an Orbifold,” hep-th/9512219; A. Sen, “M-theory on $(K3 \times S_1)/\mathbf{Z}_2$,” hep-th/9602010; A. Kumar and K. Ray, “M-theory on Orientifolds of $K3 \times S_1$,” hep-th/9602144.
- [8] P. Townsend, “D-branes from M-branes,” hep-th/95012062; A. Strominger, “Open P-Branes,” hep-th/9512059; K. Becker and M. Becker, “Boundaries in M-theory,” hep-th/9602071.
- [9] C. Hull, “String Dynamics at Strong Coupling,” hep-th/9512181.
- [10] C. Vafa, “Evidence for F-theory,” hep-th/9602022; D. Morrison and C. Vafa, “Compactification of F-theory and C-Y Manifolds,” hep-th/9602114, hep-th/9603161.
- [11] D. Kutasov and E. Martinec, “New Principles for String/Membrane Unification,” hep-th/9602049; D. Kutasov, E. Martinec, and M. O’Loughlin, “Vacua of M-theory and N=2 Strings,” hep-th/9603116.
- [12] J. Polchinski, “Dirichlet-Branes and Ramond-Ramond Charges,” hep-th/9510017.
- [13] C. Vafa, “Gas of D-Branes and Hagedorn Density of BPS States,” hep-th/9511026, “Instantons on D-branes,” hep-th/9512078; M. Bershadsky, V. Sadov, and C. Vafa, “D-Branes and Topological Field Theories,” hep-th/9511222.
- [14] A. Sen, “A Note on Marginally Stable Bound States in Type II String Theory” hep-th/9510229; “U Duality and Intersecting D-Branes,” hep-th/9511026; “T-Duality of p-Branes,” hep-th/9512062.

- [15] E. Witten, “Bound States of Strings and p -branes,” Nucl. Phys. **B460** (1996) 335, hep-th/9510135.
- [16] E. Bergshoeff, E. Sezgin, and P. Townsend, “Supermembranes and Eleven Dimensional Supergravity,” Phys. Lett. **B189** (1987) 75;
- [17] R. Dijkgraaf, E. Verlinde and H. Verlinde, “BPS Spectrum of the Five-Brane and Black Hole Entropy,” hep-th/9603126.
- [18] For an extensive set of references see A. Salam and E. Sezgin, “Supergravities in Diverse Dimensions,” North-Holland, World-Scientific (1989).
- [19] C. Callan, J. Harvey, and A. Strominger, Nucl. Phys. **B 367** (1991) 60; D. M. Kaplan and J. Michelson, “Zero-modes for the 11-dimensional membrane and five-brane,” hep-th/9601053
- [20] J. Hoppe, “Quantum Theory of a Relativistic Surface,” Ph.D. Thesis, MIT (1982). B. de Wit, J. Hoppe and H. Nicolai, “On the Quantum Mechanics of Supermembranes,” Nucl. Phys. **B305** (1988) 545–581.
- [21] M. J. Duff and J. X. Lu, “Black and Super p -Branes in Diverse Dimensions,” Nucl. Phys. **B416** (1994) 301; E. Witten, “Some Comments on String Dynamics,” hep-th/9510135; A. Strominger, “Open P-Branes,” hep-th/9512059.
- [22] M. Duff, R. Minasian, and E. Witten, “Evidence for Heterotic/Heterotic Duality,” hep-th/9601036.